A Theoretical Investigation of MHD Channel Entrance Flows

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A momentum-integral boundary-layer analysis is made of the entrance MHD flow of an incompressible electrically conducting fluid in a two-dimensional constant area channel. It is assumed that a uniform magnetic field perpendicular to the insulated channel walls exists in the channel. An electric field, perpendicular to both the magnetic field and mean flow, is permitted. The magnetic Reynolds number is assumed small. Closed-form solutions are obtained for both the laminar and turbulent cases. Variation of the freestream velocity in the flow direction is fully accounted for. Two cases are considered: the uniform velocity entry case, and the situation where the velocity profile at the channel entry plane is nonmagnetically fully developed. Agreement between the present work and more exact numerical solutions for a few particular flow cases is excellent. Increasing the Hartmann number is shown to reduce entrance lengths appreciably, except in the very low Hartmann number range. Flows with nonmagnetically fully developed entry profiles, in general, require much larger entrance lengths than flows with a uniform entry profile.

Nomenclature

a = half-height of channel
B = magnetic induction
C = const(= 0.0456)
E = electric field
f = friction factor ($\equiv 8\tau_w/\rho U_m^2$)
j = electric current density
M = Hartmann number [$\equiv (\sigma/\mu)^{1/2}Ba$]
p = static pressure
Re = Reynolds number ($\equiv 4aU_m\rho/\mu$)
U, V = velocities in x and y directions, respectively
W = half-width of channel
x, y, z = coordinates (see Fig. 1)
δ = boundary-layer thickness
δ^* = displacement thickness
ρ = fluid density
σ = fluid electric conductivity
θ = momentum thickness
μ = fluid viscosity
τ_w = wall shear stress

Subscripts

FD = fully developed
∞ = freestream
w = wall
m = mean

Introduction

A BETTER knowledge of the velocity fields, boundary-layer development, and friction factors in the entrance region of MHD channels is of interest both to the curious fluid dynamicist and the practical designer. In order to avoid the added difficulties that are present in the solution of the problem of a channel with an arbitrary, finite aspect ratio, this investigation, as most that preceded it, is addressed to the special case of an infinite aspect ratio channel, i.e., the flow between two parallel plates. It can be expected that the solutions to this problem will be applicable to high aspect ratio channels and will possibly indicate trends even when the ratio is low. In his analysis of the laminar flow case, Shercliff[1] linearized the problem by the Rayleigh approximation and obtained solutions for the deviation of the local from the ultimate, or Hartmann, velocity profile in terms of an orthogonal series. In some special cases, the leading terms of the series were evaluated. Shohet[2] and Dix[3] have independently obtained numerical solutions to the laminar flow situation when the velocity across the channel entrance is uniform. Also, for uniform entrance velocity, Moffatt[4] has obtained closed-form laminar and turbulent solutions, with the further restriction that the freestream velocity be constant. Moffatt used the momentum integral method with assumed similar velocity profiles across the boundary layers. In the present investigation, Moffatt's method has been extended to include variation of the freestream velocity in the flow direction, thereby satisfying over-all conservation of mass for a constant area channel. Two cases are treated: the uniform entry velocity case and the heretofore unsolved situation of a nonmagnetically fully developed velocity profile at the channel entry plane.

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Problem and Solutions

The geometric configuration of the problem to be solved is shown in Fig. 1. The channel is assumed to have a sufficiently high aspect ratio and negligible three-dimensional effects for the problem to be considered two dimensional. For $x \geq 0$, the flow is in the presence of uniform, mutually perpendicular, magnetic and electric fields, whereas for $x < 0$, the fields are zero. Fluid density, viscosity, and electrical conductivity are assumed to be constant for simplicity. Hall effects and induced magnetic fields are assumed to be small enough to be neglected.

With the usual boundary-layer assumptions, the integral method leads to the balance of forces and momentum fluxes:

$$\tau_w + \delta\frac{\partial p}{\partial x} - \int_0^\delta jBdy = U_\infty\frac{\partial}{\partial x}\int_0^\delta \rho U dy - \frac{\partial}{\partial x}\int_0^\delta \rho U^2 dy \tag{1}$$

The freestream momentum equation

$$\frac{dp}{dx} = -\rho U_\infty\frac{dU_\infty}{dx} + j_\infty B \tag{2}$$

and Ohm's law

$$j = \sigma(E - UB) \tag{3}$$

provide the means for expressing the pressure gradient and the Lorentz force in terms of velocities and the magnetic

Table 1 Summary of solutions

Type of flow → Entry velocity profile	Laminar	Turbulent
Uniform	$\frac{x}{aRe} = \frac{3}{20(2M^2 - \frac{2}{3} - \frac{3}{2}M^4)} \Big\{ -\left(\frac{16}{3} + M^2\right) \ln\left(1 - \frac{\delta}{3a}\right) - \frac{(1 - \frac{3}{2}M^2)\delta/a}{1 - (\delta/3a)} - \left(11M^2 + \frac{14}{3}\right) \frac{1}{2(6)^{1/2}M} \ln\frac{(6)^{1/2} + M(\delta/a)}{(6)^{1/2} - M(\delta/a)} + \left(\frac{8}{3} + \frac{M^2}{2}\right) \ln[1 - (M^2/6)\delta^2/a^2] \Big\}$ (A)	$\frac{x}{aRe} = \frac{23}{576} \frac{(\delta/a)^{9/4}}{0.06448Re^{3/4} - M^2(\delta/a)^{5/4}} - \frac{7}{45M^2} \ln\left[1 - \frac{M^2(\delta/a)^{5/4}}{0.06448Re^{3/4}}\right]$ (C)
Nonmagnetically fully developed	$\frac{x}{aRe} = \frac{3}{20(2M^2 - \frac{2}{3} - \frac{3}{2}M^4)} \Big\{ -\left(\frac{16}{3} + M^2\right) \ln\frac{1 - (\delta/3a)}{\frac{2}{3}} - \frac{\frac{3}{2}[(\delta/a) - 1](1 - \frac{3}{2}M^2)}{1 - (\delta/3a)} - \frac{(11M^2 + \frac{14}{3})}{2(6)^{1/2}M} \ln\frac{[(6)^{1/2} + (\delta/a)M][(6)^{1/2} - M]}{[(6)^{1/2} - (\delta/a)M][(6)^{1/2} + M]} + \left(\frac{8}{3} + \frac{M^2}{2}\right) \ln\frac{1 - (M^2/6)(\delta^2/a^2)}{1 - (M^2/6)} \Big\}$ (B)	$\frac{x}{aRe} = \frac{23}{576} [(\delta/a) - 1] \Big\{ \frac{(\delta/a)^{5/4}}{0.06448Re^{3/4} - M^2(\delta/a)^{5/4}} + \frac{1}{0.06448Re^{3/4} - M^2} \Big\} - \frac{7}{45M^2} \ln\frac{0.06448Re^{3/4} - M^2(\delta/a)^{5/4}}{0.06448Re^{3/4} - M^2}$ (D)

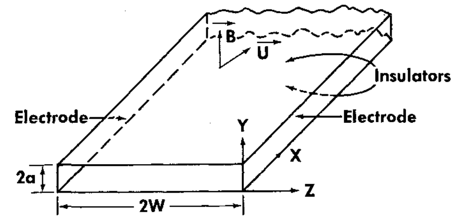

Fig. 1 Geometric configuration of the problem.

field. The resulting equation for the shear stress τ_w is then given by

$$\tau_w = \delta B^2 U_\infty \sigma \left[1 - \int_0^1 \frac{U}{U_\infty} d\left(\frac{y}{\delta}\right)\right] + \delta^* \rho U_\infty \frac{dU_\infty}{dx} + \frac{d}{dx}(\rho U_\infty^2 \theta) \quad (4)$$

wherein θ and δ^* are the momentum and displacement thicknesses, respectively.

For the laminar flow case, one can substitute $\mu(\partial U/\partial y)_w$ for the shear, which is evaluated from the assumed velocity profile, whereas for the turbulent flow, Blasius' friction law is applicable. In order to solve the equations for the boundary-layer thickness, it is assumed that in the laminar case the velocity profile consists of a parabolic variation

$$U/U_\infty = 1 - [(y/\delta) - 1]^2$$

across the boundary layer and a uniform core velocity. In the turbulent case, a one-seventh power velocity distribution in the boundary layer is assumed. The core velocity is allowed to vary in the x direction in order to satisfy the conservation of mass equation applied across the entire channel cross section. The boundary conditions at $x = 0$ are: $\delta = a$ for the nonmagnetically fully developed entry, and $\delta = 0$ for the uniform entry velocity.

With the stated assumptions, and after lengthy but simple integrations, algebraic manipulations, and nondimensionalizing, the solutions for the laminar and turbulent cases can be written in tabular form (Table 1). Whereas Eqs. (A) and (B) represent closed-form solutions in the strict sense of the word, in deriving Eqs. (C) and (D), several unmanageable integrals were encountered, and it was necessary to assume that $\delta/8a \ll 1$ and use the approximation

$$\int_{(\delta/a)_1}^{(\delta/a)_2} \frac{(\delta/a)^{5/4} d(\delta/a)}{(C/Re^{1/4}) - (M^2/Re)(\delta/a)^{5/4}} \approx \frac{1}{2}\left[\left(\frac{\delta}{a}\right)_2 - \left(\frac{\delta}{a}\right)_1\right]\left[\frac{(\delta/a)_2^{5/4}}{(C/Re^{1/4}) - (M^2/Re)(\delta/a)_2^{5/4}} + \frac{(\delta/a)_1^{5/4}}{(C/Re^{1/4}) - (M^2/Re)(\delta/a)_1^{5/4}}\right]$$

In the laminar case, it should be noted that the nondimensional length parameter x/aRe, familiar from Schlichting's[5] nonmagnetic solution, is a function of δ/a and the Hartmann number M only. The Hartmann number can be viewed as a measure of the magnetic interaction. In the turbulent case, x/aRe depends also on the Reynolds number. For the assumed flow model, the boundary-layer development and the velocity profiles are independent of the electric field if the Hartmann and Reynolds numbers are specified.

Some specific examples of the laminar solution are shown in Fig. 2, which plots the variation of δ/a with the length parameter for two Hartmann numbers, 10 and 100, for both entrance situations. The effect of the magnetic field on the entrance flow is graphically illustrated. For the nonmagnetically fully developed entrance profile at $x = 0$, the boundary layers are assumed to join at the channel center. Proceeding into the channel, a core flow develops, and the boundary layers spread apart and diminish to an asymptotic value.

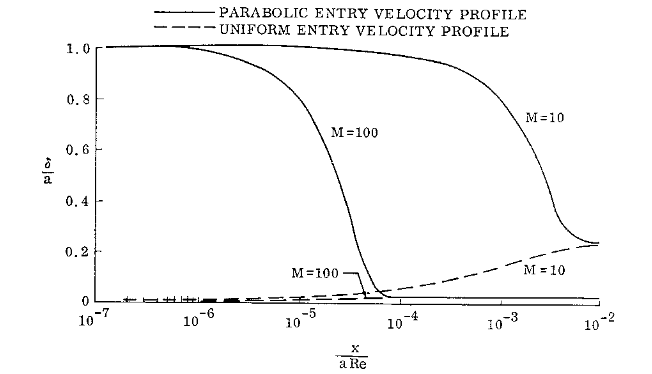

Fig. 2 Development of laminar MHD boundary layers in the channel entrance region.

For the uniform velocity entry, the boundary layer grows from zero until it attains the same asymptotic value.

The variation of the entrance length with Hartmann number is presented in Fig. 3. The entrance length is defined here as the distance required for the friction factor to come within 10% of the final, fully developed value. Generally, the results demonstrate a decrease in the entrance length with increasing Hartmann number. The curves for the nonmagnetically fully developed entrance condition show an increase in entrance length with increasing Hartmann number for low Hartmann numbers; this is consistent with the fact that the entrance length for this condition must go to zero for $M = 0$. It is interesting that the curves for the uniform velocity entrance condition must also show such a reverse trend for sufficiently low M, since the entrance lengths for nonmagnetic ($M = 0$) flow can be shown to be less than the maximum entrance length values exhibited by the curves of Fig. 3. It was impossible to use our integral technique for M values less than those yielding $\delta/a = 1$ for

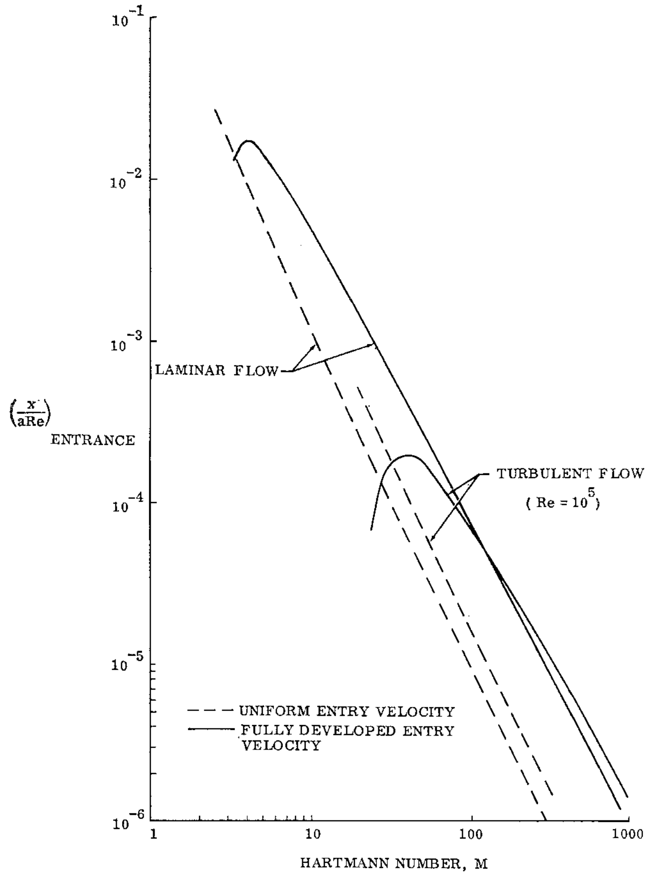

Fig. 3 MHD entrance lengths.

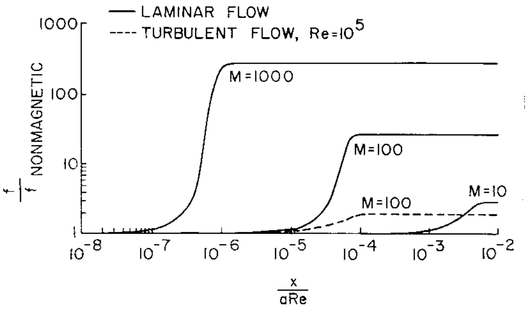

Fig. 4 Friction factor development in the channel entrance region for nonmagnetically fully developed entry velocity profiles.

fully developed flow.‡ For laminar flow, this minimum value of M was about 2.45; for the turbulent case it depended on Re and was about 19 for an Re of 10^5.

In Fig. 3, the turbulent entrance lengths generally exceed those for laminar flow. This fact is in contrast to what one would expect, since for nonmagnetic flow the reverse is true. However, over much of the Hartmann number range of Fig. 3, the entrance lengths for both laminar and turbulent flow are so small as to be almost negligible. Furthermore, the possibility of transition from turbulent to laminar flow at higher Hartmann numbers (above 110 for $Re = 10^5$ according to Ref. 6) has not been taken into consideration.

Shercliff had defined the entrance length as the distance over which the predominant term in the series, representing the deviation of the local velocity from the Hartmann profile, decreased by $1/e$. Obviously, this differs significantly from the definition used in Fig. 3. To establish a more compatible basis for comparison, the entrance length could be defined as the distance required for δ/a to reach 99% of its ultimate, fully developed value. The authors found that for the laminar flow, with uniform entrance velocity,§ such entrance lengths were within 20% of those computed by Shercliff.

The influence of the Hartmann number on the friction factor development is illustrated in Fig. 4, which contains results for the nonmagnetically fully developed entry case. For laminar flow, the increase over the nonmagnetic friction factors can amount to several orders of magnitude. For the turbulent flow, for which only one curve is shown, the increase is much smaller. The trend with Hartmann number, though not shown in Fig. 4, is similar to that for laminar flow.

Discussion

The validity of the solutions presented is contingent upon the accuracy of the assumptions made. It is important,

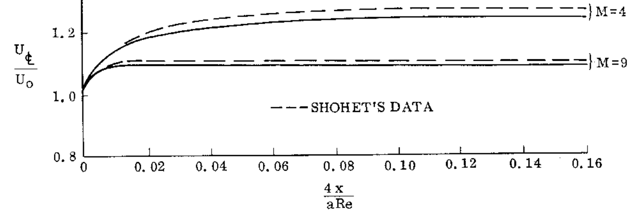

Fig. 5 Comparison of channel centerline velocities with Shohet's results (laminar flow, uniform inlet velocities U_0).

‡ It is possible to bypass this limitation in the case of uniform entrance velocity if the entrance length is redefined as the length to the position at which $\delta/a = 1$.

§ Shercliff did not work out the solution for the nonmagnetically fully developed entry case.

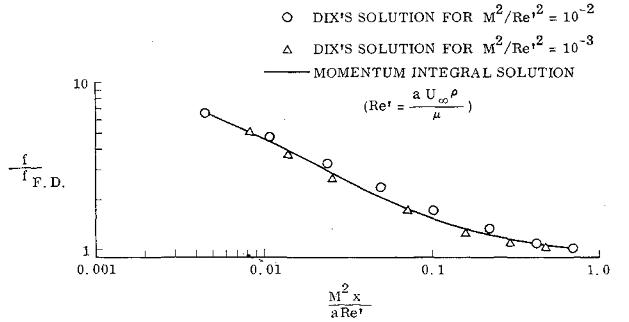

Fig. 6 Comparison of skin friction with the numerical solution of Dix (laminar flow, uniform entry velocity).

therefore, to examine these carefully. It is felt that boundary-layer assumptions would be reasonably valid up to those Hartmann numbers for which the entrance lengths are of the same order of magnitude as the asymptotic boundary-layer thickness. For larger Hartmann numbers, it would be probably inaccurate to assume that $\partial p/\partial y \ll \partial p/\partial x$ or that $V \ll U$.

There are neither experimental measurements nor exact numerical solutions of the nonmagnetically fully developed entry problem. Therefore, the support for the assumed velocity profiles can only come, by inference, from comparisons with numerical solutions to the related problem with uniform entry velocity.¶ For the laminar case, Shohet and Dix have independently obtained such solutions.

As seen in Fig. 5, the comparison of the development of core velocities with Shohet's numerical solution shows reasonably good agreement. The agreement would improve for higher Hartmann numbers. This comparison is, of course, for the uniform entry velocity case.

Comparison of friction factors with those of Dix's doctoral thesis (Fig. 6) shows an embarrassingly good agreement. The ordinate in this plot is the ratio of the local friction factor to the friction factor of the fully developed flow. Data shown are again for the case of uniform entry velocity.

It is well known that the effect of the magnetic field is to flatten the velocity profile. Comparison of the laminar velocity profiles for very large x, as obtained in the present work, shows a reasonably good agreement with Hartmann's profiles. At the channel entry, the parabolic profile for the laminar case is, of course, an exact representation of the fully developed nonmagnetic profile.

For the turbulent case, only the friction factors corresponding to x larger than the entrance length can be compared to existing data, namely, Murgatroyd's[6] test results. That the agreement is quite satisfactory is shown in Fig. 7 where the friction factors corresponding to three Reynolds numbers have been graphed. They were computed from

$$f_{FD.} = 0.446 \frac{(M/Re)^{0.4}}{\{1 - [0.0139/(M/Re)^{1.6}Re]\}^{7/4}} \qquad (5)$$

¶ The comparison with Shercliff's approximate solution was discussed in the preceding section.

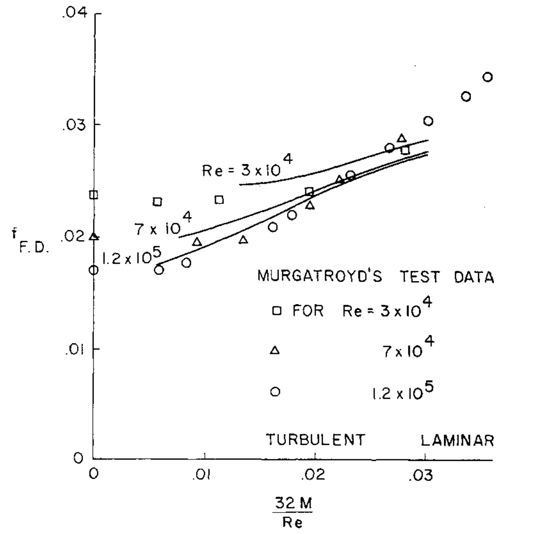

Fig. 7 Comparison of theoretical fully developed turbulent friction factors with Murgatroyd's data.

It is important to be cognizant of the restricted range of applicability for the turbulent results of the present work. The lower limits of $32M/Re$ on the theoretical curves in Fig. 7 are given by that Hartmann number for which $\delta/a = 1$ as $x \to \infty$. The upper limit is set by $32M/Re \approx 0.030$ above which Murgatroyd's data indicate that the flow becomes laminar.

It is worth noting from Fig. 7 that the turbulent, fully developed friction factor is a function not only of M/Re but also of Re itself, which is in agreement with Murgatroyd. The Reynolds number dependence, lower limit on M/Re and the much closer general agreement with test results constitute, in the authors' opinion, a substantial improvement over Moffatt's correlation.

In view of the foregoing comparisons, it is felt that the assumptions made are at least partially substantiated, and the results presented are close to reality.

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